DBJ-003-2015001 Seat No. ________

B. Sc. (Sem. V) (CBCS) (W.E.F.-2019) Examination

June – 2022

Math-5(A) : Mathematics

(Mathematical Analysis-1 & Abstract Algebra-1)

Faculty Code : 003
Subject Code : 2015001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions :

(1) Attempt any five questions.
(2) Figures to the right indicate full marks of the question.

1 (A) Answer the following questions : **4**

(1) Define : Interior point.
(2) Define : Discrete metric space.
(3) If $(\mathbb{R}, d)$ is a discrete metric space, then find $N(\pi, 1/2)$.
(4) If $(\mathbb{R}, d)$ is a usual metric space, then find $(0,3)'$.

(B) Attempt the following : **2**

(1) Show that $\mathbb{Z}$ is closed subset of $\mathbb{R}$.

(C) Attempt the following : **3**

(1) Prove that : Any finite subset of a metric space is closed.

(D) Attempt the following : **5**

(1) If (X,d) is a metric space, then show that $d_1 : X \times X \to \mathbb{R}; d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is a bounded metric on X.

2 (A) Answer the following : **4**

(1) Let $A=\left\{\frac{1}{2n} \mid n\in\mathbb{N}\right\}\subset\mathbb{R}$. Find $\overline{A}$.

(2) Define : Limit point.

(3) Let $(\mathbb{R},d)$ be a usual metric space. What is in $\mathbb{R}$?

(4) True or False : Let (X, d) be a metric space. Then ϕ is an open set in X.

(B) Attempt the following : **2**

(1) Show that $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}, d(x,y)=|\sin x-\sin y|$ is not a metric on R.

(C) Attempt the following : **3**

(1) Prove that if (X, d) is a metric space, then $|d(x,z)-d(y,z)|\le d(x,y), \forall x,y,z\in X$.

(D) Attempt the following : **5**

(1) Let X be a metric space and $A,B\subset X$. Show that $(A\cap B)^\circ = A^\circ\cap B^\circ$.

3 (A) Answer the following : **4**

(1) In usual notation define $U(P,f)$.

(2) Let $f(x)=x, x\in[0,1]$ and $P=\left\{0,\frac{1}{3},\frac{2}{3},1\right\}$ be a partition of [0,1]. Compute $U(P,f)$.

(3) Define : Norm of the partition.

(4) True or False : Every bounded function on [a,b] is R-integrable.

(B) Attempt the following : **2**

(1) State First Mean Value Theorem of Integral Calculus.

(C) Attempt the following : **3**

(1) Using second definition prove that $\int_1^2 3x\,dx=\frac{9}{2}$.

(D) Attempt the following : **5**

(1) Show that :

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}\right]=\log_e 2.$$

4 (A) Answer the following : **4**

(1) In usual notation define $L(P,f)$.

(2) State Darboux's Theorem.

(3) True or False : If $fg \in R[a,b]$, then $f \in R[a,b]$ and $g \in R[a,b]$.

(4) True or False : Let P, P^* be partitions of [a,b] such that $P \subset P^*$. Then $\|P^*\| \le \|P\|$.

(B) Attempt the following : **2**

(1) Find $L(P,f)$ and $U(P,f)$ for the function

$$f(x)=x^2, x\in[0,1] \text{ and } P=\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}.$$

(C) Attempt the following : **3**

(1) If $f \in R[a,b]$ then prove that $f^2 \in R[a,b]$.

(D) Attempt the following : **5**

(1) If $P_1, P_2 \in [a,b]$, then show that $L(P_1,f) \le U(P_2,f)$.

5 (A) Answer the following : **4**

(1) Express $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{2r}{n}+3\right)$ as definite integral.

(2) Let $f(x)=x$ for all $x\in[0,1]$. Let $P=\left\{0,\frac{1}{3},\frac{2}{3},1\right\}$ be a partition of [0,1]. Find $L(P,f)$.

(3) State second mean value theorem of integral calculus.

(4) True or False : Every R-integrable function on [a,b] is a continuous function on [a,b].

(B) Attempt the following : **2**

(1) Prove that $\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$; where f is real valued function define on [a,b].

(C) Attempt the following : **3**

(1) If $f \in R[a,b]$, then show that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a),$$

where m and M are the infimum and supermum of f on [a,b], respectively.

(D) Attempt the following : **5**

(1) For $0 < a < b$, Prove that

$$\frac{\pi^2}{2b} \le \int_0^\pi \frac{x}{a\cos^2\frac{x}{2} + b\sin^2\frac{x}{2}}\,dx \le \frac{\pi^2}{2a}$$

6 (A) Answer the following : **4**

(1) Define : Binary Operation.

(2) For $a,b \in \mathbb{Q}$, define $a*b = \frac{ab}{2}$. What is the identity element for $*$?

(3) For $a,b \in \mathbb{Q}$, define $a*b = \frac{ab}{2}$. What is the inverse of 2 ?

(4) True of False : Subtraction is a commutative binary operation on $\mathbb{Z}$.

(B) Attempt the following : **2**

(1) Let $(G,\cdot)$ be a group. Show that the identity in G is unique.

(C) Attempt the following : **3**

(1) Define cancelation laws. Show that cancelation laws holds in a group G.

(D) Attempt the following : **5**

(1) Let $M_2(\mathbb{R})$ be the set of all 2×2 matrices over $\mathbb{R}$. Show that

$$GL(2,\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid |A| \neq 0\}$$

forms a group with respect to usual matrix multiplication.

7 (A) Answer the following : **4**

(1) Define : Subgroup

(2) Define : Coset

(3) Define : Center of the group

(4) Define : The alternate group A_n

(B) Attempt the following : **2**

(1) Let $(G,\cdot)$ be a group and $a,b \in G$. Show that $(a\cdot b)^{-1} = b^{-1}\cdot a^{-1}$.

(C) Attempt the following : **3**

(1) Answer the following :

(a) Give an example of a group having exactly 8 elements.

(b) True or False : $\mathbb{Z}_6$ is a subgroup of $\mathbb{Z}_{10}$.

(c) What is the order of A_5 ?

(D) Attempt the following : **5**

(1) Let $(G,\cdot)$ be a group. Show that : If $a^2 = e$ for all $a \in G$, then G is abelian.

8 (A) Answer the following : **4**

(1) Check whether $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 1 & 3 & 6 & 4 & 2 \end{pmatrix} \in S_7$ is odd or even ?

(2) Let G be a group, $x,y \in G$ and $O(x) = 8$. If $z = yxy^{-1}$, then $O(z)$ = _____.

(3) If $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 2 & 6 & 5 & 7 \end{pmatrix} \in S_7$, then find g^{-1}.

(4) Define : Transposition.

(B) Attempt the following

(1) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 1 & 3 & 6 & 4 & 2 \end{pmatrix}$, **2**

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 2 & 6 & 5 & 7 \end{pmatrix} \in S_7$.

Find fg.

(C) Attempt the following : **3**

(1) Let H be a subgroup of a group G. Then show that

(a) If $a \in bH$, then $aH = bH$.

(b) For $a, b \in G$ show that either $aH \cap bH = \phi$ or $aH = bH$.

(D) Attempt the following : **5**

(1) Let H be a subgroup of a group G. For $a, b \in G$ define $a \equiv b \pmod{H} \Leftrightarrow ab^{-1} \in H$. Prove that $\equiv$ is an equivalence relation.

9 (A) Answer the following : **4**

(1) Define : Cyclic group.

(2) State Lagrange's Theorem.

(3) True or False : There exists a group of order 10 having a subgroup of order 4.

(4) True or False : Every cyclic group is abelian.

(B) Attempt the following : **2**

(1) Let G be a group and $a \in G$. Define $N(a)$ and show that $N(a)$ is a subgroup of the group G.

(C) Attempt the following : **3**

(1) Prove that : For $n \geq 3$, every $f \in A_n$ can be express as product of 3-cycle.

(D) Attempt the following : **5**

(1) State and prove Lagrange's Theorem.

10 (A) Answer the following : **4**

(1) Define : Normal subgroup.

(2) Define : Isomorphism.

(3) Define : Quotient group.

(4) True or False : $SL(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$.

(B) Attempt the following : **2**

(1) Show that every subgroup of an abelian group is a normal subgroup.

(C) Attempt the following : **3**

(1) Let H be a normal subgroup of a group G. For $a,b \in G$ show that $(aH)(bH) = abH$.

(D) Attempt the following : **5**

(1) State and prove Cayley's Theorem.